

CONVERGENCE TO EQUILIBRIUM FOR THE SEMILINEAR PARABOLIC EQUATION WITH DYNAMICAL BOUNDARY CONDITION

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Abstract

This paper is concerned with the asymptotic behavior of the solution to the following semilinear parabolic equation

$$u_t - \Delta u + f(u) = 0, \quad (x, t) \in \Omega \times \mathbb{R}^+,$$

subject to the dynamical boundary condition

$$\partial_\nu u + \mu u + u_t = 0, \quad (x, t) \in \Gamma \times \mathbb{R}^+,$$

and the initial condition

$$u|_{t=0} = u_0(x), \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}^+$) is a bounded domain with smooth boundary Γ , ν is the outward normal direction to the boundary and $\mu \in \{0, 1\}$. f is analytic with respect to unknown function u . Our main goal is to prove the convergence of a global solution to an equilibrium as time goes to infinity by means of a suitable Łojasiewicz-Simon type inequality.

Keywords: Semilinear parabolic equation, dynamical boundary condition, Łojasiewicz-Simon inequality.

1 Introduction

In this paper we consider the following semilinear parabolic equation

$$u_t - \Delta u + f(u) = 0, \quad (x, t) \in \Omega \times \mathbb{R}^+, \quad (1.1)$$

subject to the dynamical boundary condition

$$\partial_\nu u + \mu u + u_t = 0, \quad (x, t) \in \Gamma \times \mathbb{R}^+, \quad (1.2)$$

and the initial condition

$$u|_{t=0} = u_0(x), \quad x \in \Omega. \quad (1.3)$$

In the above, $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}^+$) is a bounded domain with smooth boundary Γ . ν is the outward normal direction to the boundary and $\mu \in \{0, 1\}$.

Parabolic equation and systems with dynamical boundary conditions have been extensively studied in the literature (see for instance, [3, 4, 7–9, 12, 15, 19] etc.). In particular, local existence and uniqueness of solution to general quasilinear parabolic equation (systems) with dynamical boundary condition has been established in a series of papers by Escher [7–9] (see also [15] for a semigroup approach in the $H_p^2(\Omega)$ -setting, and [4] for the solvability result in a weighted Hölder space). Moreover, in [8, 9], the author showed that the solution defines a local semiflow on certain Bessel potential spaces including the standard Sobolev space $W^{1,p}$. Based on the approach in [7–9] etc., in [12] the authors studied the wellposedness of a semilinear parabolic equation subject to a nonlinear dynamical boundary condition with subcritical growth for nonlinearities and then investigated the large time behavior of the solution such as blow-up phenomenon. In our present paper, we are now interested in the question whether the global solution to a class of semilinear parabolic equation with subcritical nonlinearity and dynamical boundary condition (1.1)–(1.3) will converge to an equilibrium as time goes to infinity.

Before stating our main results we make some assumptions on nonlinearity f .

(F1) $f(s)$ is analytic for $s \in \mathbb{R}$.

(F2)

$$|f'(s)| \leq c(1 + |s|^p), \quad \forall s \in \mathbb{R}, \quad p \in [0, \alpha),$$

where

$$\alpha := \begin{cases} +\infty & n = 1, 2, \\ \frac{4}{n-2} & n \geq 3. \end{cases}$$

(F3)

- for $\mu = 1$,

$$\liminf_{|s| \rightarrow +\infty} \frac{f(s)}{s} > -\frac{1}{4}\lambda,$$

where $\lambda > 0$ is the best Sobolev constant in the following imbedding inequality

$$\int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} u^2 dS \geq \lambda \int_{\Omega} u^2 dx;$$

- for $\mu = 0$,

$$\liminf_{|s| \rightarrow +\infty} \frac{f(s)}{s} > 0.$$

Remark 1.1. *Assumption (F1) is made so that we can use an extended Łojasiewicz-Simon inequality to prove our convergence result (see Theorem 1.2 below). Assumption (F2) implies that the nonlinear term has a subcritical growth. Under this assumption, we are able to prove the local existence and uniqueness of solutions by adopting the argument in [12] with some modifications. Moreover, due to the subcritical growth, existence of a strong solution to stationary problem can be obtained by variational method. Assumption (F3) is a condition needed to ensure global existence of solution to our problem (1.1)–(1.3). For simplicity of exposition the nonlinear term f is assumed to depend only on u . However, the results in this paper still remain true for the nonlinear term $f(x, u)$ with additional smoothness assumption with respect to x .*

Throughout this paper we simply denote the norm in $L^2(\Omega)$ by $\|\cdot\|$ and equip $H^1(\Omega)$ with the equivalent norm

$$\|u\|_{H^1(\Omega)} = \left(\int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} u^2 dS \right)^{1/2}.$$

We are now in a position to state our main results.

Theorem 1.1. *Suppose (F1)(F2)(F3) are satisfied. Then for any initial data $u_0 \in H^1(\Omega)$, problem (1.1)–(1.3) admits a unique global solution such that*

$$u \in C([0, +\infty); H^1(\Omega)) \cap C((0, +\infty); H^2(\Omega)) \cap C^1((0, +\infty); L^2(\Omega)),$$

$$\gamma(u) \in C([0, +\infty); H^{\frac{1}{2}}(\Gamma)) \cap C((0, +\infty); H^{\frac{3}{2}}(\Gamma)) \cap C^1((0, +\infty); H^{\frac{1}{2}}(\Gamma)),$$

where $\gamma(u)$ is the trace operator. Moreover, if $u_0 \in H^2(\Omega)$, then

$$u \in C([0, +\infty); H^2(\Omega)), \quad \gamma(u) \in C([0, +\infty); H^{\frac{3}{2}}(\Gamma)).$$

Theorem 1.2. *Suppose that (F1) (F2) (F3) are satisfied. Then for any initial data $u_0 \in H^1(\Omega)$, and in addition $u_0 \in L^{\frac{4n}{n-2}}(\Omega)$, $\gamma(u_0) \in L^{\frac{4n}{n-2}}(\Gamma)$ when $n \geq 3$, the global solution to problem (1.1)–(1.3) converges to an equilibrium $\psi(x)$ in the topology of $H^1(\Omega)$ as time goes to infinity, i.e.,*

$$\lim_{t \rightarrow +\infty} (\|u(t, \cdot) - \psi\|_{H^1(\Omega)} + \|u_t(t, \cdot)\|) = 0. \quad (1.4)$$

Here $\psi(x)$ is an equilibrium to problem (1.1)–(1.3), i.e., $\psi(x)$ is a strong solution to the following nonlinear elliptic boundary value problem:

$$\begin{cases} -\Delta\psi + f(\psi) = 0, & x \in \Omega, \\ \partial_\nu\psi + \mu\psi = 0, & x \in \Gamma. \end{cases} \quad (1.5)$$

Before giving the detailed proof of our convergence theorem, we briefly recall some related results in the literature. For the semilinear parabolic equation with homogeneous Dirichlet boundary condition, concerning convergence to an equilibrium as time goes to infinity, we notice that in one spacial dimension, the situation is much simpler: the set of stationary states is discrete and one can take advantage of existence of a Lyapunov functional to prove that any bounded global solution will converge to an equilibrium (see e.g. [22, 33]). In the case of higher spatial dimension, it was proved in [17, 26] by use of the so-called Łojasiewicz-Simon inequality that any bounded global solution will converge to a single stationary state provided that f is real analytic. This kind of result is highly nontrivial, since in higher spatial dimension the stationary states can form a continuum (see, e.g., [13]). Moreover, a counterexample for semilinear parabolic equations with C^∞ nonlinearities are given in the literature (see [24], also [23]) to show that there is a bounded global solution whose ω -limit set is diffeomorphic to the unit circle S^1 . After this breakthrough, many further contributions were made for various types of nonlinear evolution equations (see, e.g., [2, 10, 11, 14, 16, 18, 25, 29–32] and references therein).

Some new features of our present work are as follows. First, due to the presence of dynamical term u_t , it turns out that for the corresponding elliptic operator, the dissipative boundary condition should be considered as non-homogeneous. As a result, the Łojasiewicz-Simon type inequality we are going to derive is naturally different from the one with the usual homogeneous boundary conditions in the literature. Furthermore, we are able to treat both non-homogenous Neumann and Robin boundary conditions (corresponding to $\mu = 0, 1$, respectively). Second, the subcritical growth of nonlinearity

seems natural as stated in Remark 1.1. Besides, we do not have any restriction on spatial dimension. Thus, our result improves the growth condition of nonlinear term stated in [16, Chapter 3, Section 3.2] even for the Dirichlet boundary condition.

The remaining part of this paper is organized as follows. In Section 2 we prove the existence and uniqueness of the global solution (Theorem 1.1) and derive some uniform *a priori* estimates. In Section 3 we first study the stationary problem. Then we proceed to establish the generalized Łojasiewicz–Simon type inequality, and to complete the proof of Theorem 1.2.

2 Global Existence and Uniqueness

Let

$$\begin{aligned} E &:= \{(u, v)^T \in H^2(\Omega) \times H^{\frac{3}{2}}(\Gamma) : \gamma(u) = v\}; \\ E_1 &:= \{(u, v)^T \in H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma) : \gamma(u) = v\}; \\ F &:= L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma). \end{aligned}$$

Following [7–9, 12] we can rewrite problem (1.1)–(1.3) in the following form:

$$\begin{cases} \frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} + A \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u) \\ 0 \end{pmatrix} = 0, \\ (u(0), v(0))^T = (u_0, \gamma(u_0))^T, \end{cases} \quad (2.1)$$

with

$$A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Delta u \\ \mu v + \gamma(\partial_\nu u) \end{pmatrix},$$

and E is the domain of A , i.e., $D(A) = E$.

By the results in Escher [9] and Fila & Quittner [12], we have

Theorem 2.1. *Let $u_0 \in H^1(\Omega)$ and assumptions **(F1)****(F2)** be satisfied. Then problem (1.1)–(1.3) admits a unique maximal solution $u(t)$ such that*

$$Z := (u, \gamma(u))^T \in C([0, T_{\max}); E_1) \cap C((0, T_{\max}); E) \cap C^1((0, T_{\max}); F).$$

Moreover, the solution $u(t)$ exists globally if $u(t)$ is bounded in $H^1(\Omega)$. In addition, if $u_0 \in H^2(\Omega)$, then $Z \in C([0, T_{\max}); E) \cap C^1([0, T_{\max}); F)$.

Proof. Let $u_0 \in H^1(\Omega)$. By assumption **(F2)**, the condition (W) in Escher [9] is satisfied. Thus $Z \in C([0, T_{max}); E_1)$ follows from [9, Theorem 1]. On the other hand, it follows from [12, Theorem 2.2] that $Z \in C((0, T_{max}); E) \cap C^1((0, T_{max}); F)$ and furthermore $Z \in C([0, T_{max}); E) \cap C^1([0, T_{max}); F)$, if $u_0 \in H^2(\Omega)$. \square

Define

$$E(u(t)) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{\mu}{2} \int_{\Gamma} u^2 dS + \int_{\Omega} F(u(t)) dx \quad (2.2)$$

where $F(z) = \int_0^z f(s) ds$.

It's easy to see that $\forall t \in (0, T_{max})$,

$$\frac{d}{dt} E(u(t)) + \|u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 = 0, \quad (2.3)$$

which implies that $E(u)$ is decreasing respect to time.

Next we prove the global existence of the solution to problem (1.1)–(1.3).

Theorem 2.2. *Assume **(F1)**(**F2**)(**F3**) hold. Then the solution $u(t)$ obtained in Theorem 2.1 exists globally, i.e., $T_{max} = +\infty$. Furthermore, assuming in addition that $u_0 \in L^{\frac{4n}{n-2}}(\Omega)$, $\gamma(u_0) \in L^{\frac{4n}{n-2}}(\Gamma)$ when $n \geq 3$, then for any $\sigma > 0$ we have the following uniform estimate*

$$\|u\|_{H^2(\Omega)} \leq C_{\sigma}, \quad \forall t \geq \sigma,$$

where C_{σ} is a positive constant depending only on $\|u_0\|_{H^1(\Omega)}$, $\|u_0\|_{L^{\frac{4n}{n-2}}(\Omega)}$, $\|\gamma(u_0)\|_{L^{\frac{4n}{n-2}}(\Gamma)}$, σ and $|\Omega|$.

Proof. By Theorem 2.1, in order to get the global existence, it suffices to obtain uniform *a priori* estimates on $\|u\|_{H^1(\Omega)}$.

From the Sobolev imbedding theorem and the growth assumption **(F2)**, we can get

$$\begin{aligned} \int_{\Omega} F(u) dx &= \int_{\Omega} \int_0^u f(s) ds \\ &\leq C \int_{\Omega} \int_0^{|u|} (1 + |s|^{p+1}) ds dx \leq C \int_{\Omega} (|u| + |u|^{p+2}) dx \\ &\leq C(\|u\|_{L^1(\Omega)} + \|u\|_{L^{p+2}(\Omega)}^{p+2}) \leq C(\|u\|_{H^1(\Omega)}). \end{aligned} \quad (2.4)$$

This implies

$$E(u(t)) \leq C(\|u\|_{H^1(\Omega)}). \quad (2.5)$$

In the same way as in [5], we can also get an estimate in the opposite direction.

To see this, for sufficiently small $\delta > 0$, we consider two cases.

Case 1: $\mu = 1$.

From the assumption **(F3)**, there exists $N = N(\delta)$ being a positive constant such that $f(z)/z \geq -\lambda + 2\delta$ for $|z| \geq N$. Then we have

$$F(s) = \int_0^N \frac{f(z)}{z} z dz + \int_N^s \frac{f(z)}{z} z dz \geq -\frac{\lambda - \delta}{2} s^2 \quad (2.6)$$

for $|s| \geq (\frac{1}{\delta}(N^2 - 2C))^{1/2} := M$, where $C = \int_0^N \frac{f(z)}{z} z dz$.

For negative s one can repeat the same computation with N replaced by $-N$. Now we have

$$\int_{\Omega} F(u) dx = \int_{|u| \leq M} F(u) dx + \int_{|u| > M} F(u) dx \geq -\frac{\lambda - \delta}{2} \int_{\Omega} u^2 dx + C(|\Omega|, f) \quad (2.7)$$

where $C(|\Omega|, f) = |\Omega| \min_{|s| \leq M} F(s)$.

Thus we can deduce that

$$E(u(t)) \geq \varepsilon \left(\frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Gamma} u^2 dS \right) + C(|\Omega|, f) \quad (2.8)$$

provided $\varepsilon \leq \frac{\delta}{2\lambda}$. Taking $C = 1/\varepsilon$, we get

$$\frac{1}{2} \|u\|_{H^1(\Omega)}^2 = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Gamma} u^2 dS \leq C(E(u(t)) - C(|\Omega|, f)). \quad (2.9)$$

Case 2: $\mu = 0$.

By assumption **(F3)**, there exists $N = N(\delta)$ being a positive constant such that $f(z)/z \geq 2\delta$ for $|z| \geq N$. Then we have

$$F(s) = \int_0^N \frac{f(z)}{z} z dz + \int_N^s \frac{f(z)}{z} z dz \geq \frac{\delta}{2} s^2 \quad (2.10)$$

for $s^2 \geq \frac{1}{\delta}(N^2 - 2C)$. Similar to Case 1, we have

$$\int_{\Omega} F(u) dx = \int_{|u| \leq M} F(u) dx + \int_{|u| > M} F(u) dx \geq \frac{\delta}{2} \int_{\Omega} u^2 dx + C(|\Omega|, f) \quad (2.11)$$

where $C(|\Omega|, f) = |\Omega| \min_{|s| \leq M} F(s) - \frac{\delta}{2} M^2 |\Omega|$.

Thus we can deduce that

$$E(u(t)) \geq \left(\frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{\delta}{2} \int_{\Omega} u^2 dx \right) + C(|\Omega|, f) \quad (2.12)$$

Taking $C = 1/\delta$, we get

$$\frac{1}{2} \|u\|_{H^1(\Omega)}^2 = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} u^2 dx \leq C(E(u(t)) - C(|\Omega|, f)). \quad (2.13)$$

For both cases, we have

$$\begin{aligned}\|u\|_{H^1(\Omega)}^2 &\leq C(E(u(t)) - C(|\Omega|, f)) \leq C(E(u_0) - C(|\Omega|, f)) \\ &\leq C(\|u_0\|_{H^1(\Omega)}) - C(|\Omega|, f)\end{aligned}$$

This implies the following uniform estimate:

$$\|u\|_{H^1(\Omega)} \leq M, \quad \forall t \in (0, T_{max}) \quad (2.14)$$

where M does not depend on T_{max} . According to Theorem 2.1, this leads to $T_{max} = +\infty$, i.e., the solution exists globally.

In order to get the uniform bound of $u(t)$ in $H^2(\Omega)$ norm, we are going to get some higher-order estimates via a formal argument. However, this procedure can be made rigorously within an appropriate regularization scheme (e.g. [35], Chapter 6).

In what follows we only consider the case $n \geq 3$ since the cases $n = 1, 2$ we can get the same result with a simpler proof (see Remark 2.1).

Again we consider two cases.

Case 1 $\mu = 1$. Multiplying (1.1) by $|u|^{p-2}u$, integrating over Ω , we get

$$\frac{1}{p} \frac{d}{dt} \left(\int_{\Omega} |u|^p dx + \int_{\Gamma} |u|^p dS \right) + (p-1) \int_{\Omega} |u|^{p-2} |\nabla u|^2 dx + \int_{\Omega} f(u) u |u|^{p-2} dx + \int_{\Gamma} |u|^p dS = 0 \quad (2.15)$$

Taking $p = \frac{4n}{n-2}$, from assumption **(F3)**, we can deduce that for $\delta > 0$ sufficiently small there exists $N = N(\delta) > 0$ such that

$$\frac{f(u)}{u} \geq \left(-\frac{4(p-1)}{p^2} \lambda + \delta \lambda \right) \quad \text{for } |u| \geq N, \quad n \geq 3,$$

and as a result

$$f(u)u \geq \left(-\frac{4(p-1)}{p^2} \lambda + \delta \lambda \right) u^2 \quad \text{for } |u| \geq N. \quad (2.16)$$

(2.16) implies that

$$\int_{\Omega} f(u)u |u|^{p-2} dx \geq C_f + \left(-\frac{4(p-1)}{p^2} \lambda + \delta \lambda \right) \int_{|u| \geq N} |u|^p dx \quad (2.17)$$

where $C_f = |\Omega| \min_{|s| \leq N} f(s)s |s|^{p-2}$. Then we get

$$\begin{aligned}&\frac{1}{p} \frac{d}{dt} \left(\int_{\Omega} |u|^p dx + \int_{\Gamma} |u|^p dS \right) + \frac{4(p-1)}{p^2} \int_{\Omega} |\nabla |u|^{\frac{p}{2}}|^2 dx + \int_{\Gamma} |u|^p dS \\ &\leq \left(\frac{4(p-1)}{p^2} \lambda - \delta \lambda \right) \int_{\Omega} |u|^p dx + C.\end{aligned} \quad (2.18)$$

Using assumption **(F3)** and applying the imbedding inequality

$$\int_{\Omega} |\nabla w|^2 dx + \int_{\Gamma} w^2 dS \geq \lambda \int_{\Omega} w^2 dx$$

to $|u|^{p/2}$, we get

$$\frac{d}{dt} \left(\int_{\Omega} |u|^p dx + \int_{\Gamma} |u|^p dS \right) + C_1 \left(\int_{\Omega} |u|^p dx + \int_{\Gamma} |u|^p dS \right) \leq C_2. \quad (2.19)$$

Integrating with respect to t yields that

$$\int_{\Omega} |u|^p dx + \int_{\Gamma} |u|^p dS \leq e^{-C_1 t} \left(\int_{\Omega} |u_0|^p dx + \int_{\Gamma} |u_0|^p dS \right) + \frac{C_2}{C_1}. \quad (2.20)$$

Thus, we get

$$\int_{\Omega} |u|^{\frac{4n}{n-2}} dx \leq C \quad (2.21)$$

which implies

$$\|f'(u)\|_{L^n(\Omega)} \leq C. \quad (2.22)$$

Here C is a constant depending on $\|u_0\|_{L^{\frac{4n}{n-2}}(\Omega)}$, $\|\gamma(u_0)\|_{L^{\frac{4n}{n-2}}(\Gamma)}$ and $|\Omega|$.

Differentiating (1.1) with respect to t , we get

$$u_{tt} - \Delta u_t + f'(u)u_t = 0. \quad (2.23)$$

Multiplying (2.23) by u_t and integrating over Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} (\|u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2) + \|\nabla u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 + \int_{\Omega} f'(u)u_t^2 dx = 0. \quad (2.24)$$

It follows from Hölder's inequality and the Gagliardo-Nirenberg inequality that

$$\begin{aligned} \left| \int_{\Omega} f'(u)u_t^2 dx \right| &\leq \|f'(u)\|_{L^n} \|u_t\|_{L^{\frac{2n}{n-1}}}^2 \\ &\leq C \|f'(u)\|_{L^n} (\|\nabla u_t\| \|u_t\| + \|u_t\|^2) \\ &\leq \frac{1}{2} \|\nabla u_t\|^2 + C \|u_t\|^2. \end{aligned} \quad (2.25)$$

Hence, we have

$$\begin{aligned} &t \left(\|u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 \right) + \int_0^t \tau \left(\|\nabla u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 \right) d\tau \\ &\leq C \int_0^t \tau \|u_t\|^2 d\tau + \int_0^t \left(\|u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 \right) d\tau. \end{aligned} \quad (2.26)$$

From (2.3) and (2.5) we can conclude that for any $t > 0$

$$t \left(\|u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 \right) \leq C(1+t), \quad \int_0^t \tau \left(\|\nabla u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 \right) d\tau \leq C(1+t). \quad (2.27)$$

Multiplying (2.23) by $-\Delta u_t$ and integrating over Ω , then from the boundary condition and Hölder's inequality we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\nabla u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 \right) + \|u_{tt}\|_{L^2(\Gamma)}^2 + \|\Delta u_t\|^2 \\ & \leq \|f'(u)\|_{L^n(\Omega)} \|u_t\|_{L^{\frac{2n}{n-2}}(\Omega)} \|\Delta u_t\|. \end{aligned} \quad (2.28)$$

By (2.22) and the Sobolev imbedding theorem, (2.28) yields that

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 \right) + \|\Delta u_t\|^2 + \|u_{tt}\|_{L^2(\Gamma)}^2 \leq \frac{1}{2} \|\Delta u_t\|^2 + C \|u_t\|_{H^1(\Omega)}^2 \quad (2.29)$$

Multiplying (2.29) by t^2 and integrating from 0 to t ,

$$\begin{aligned} & t^2 \left(\|\nabla u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 \right) + \int_0^t \tau^2 \|\Delta u_t\|^2 d\tau + 2 \int_0^t \tau^2 \|u_{tt}\|_{L^2(\Gamma)}^2 d\tau \\ & \leq \int_0^t \tau \|\nabla u_t\|^2 d\tau + \int_0^t \tau \|u_t\|_{L^2(\Gamma)}^2 d\tau + 2C \int_0^t \tau^2 \|u_t\|_{H^1(\Omega)}^2 d\tau. \end{aligned} \quad (2.30)$$

Combining it with (2.27) yields that

$$t^2 \|u_t\|_{H^1(\Omega)}^2 \leq C(1+t^2), \quad \forall t > 0. \quad (2.31)$$

Case 2 $\mu = 0$.

Similarly we have

$$\frac{1}{p} \frac{d}{dt} \left(\int_{\Omega} |u|^p dx + \int_{\Gamma} |u|^p dS \right) + (p-1) \int_{\Omega} |u|^{p-2} |\nabla u|^2 dx + \int_{\Omega} f(u) u |u|^{p-2} dx = 0, \quad (2.32)$$

where $p = \frac{4n}{n-2}$. From **(F3)**, for $\delta > 0$ small there exists $N = N(\delta) > 0$ such that

$$f(u)u \geq \delta u^2 \quad \text{for } |u| \geq N, \quad (2.33)$$

which implies

$$\int_{\Omega} f(u) u |u|^{p-2} dx \geq C_f + \delta \int_{|u| \geq N} |u|^p dx \geq (C_f - \delta N^p |\Omega|) + \delta \int_{\Omega} |u|^p dx, \quad (2.34)$$

where $C_f = |\Omega| \min_{|s| \leq N} f(s) s |s|^{p-2}$. Then we get

$$\frac{d}{dt} \left(\int_{\Omega} |u|^p dx + \int_{\Gamma} |u|^p dS \right) + C \int_{\Omega} \left(|\nabla |u|^{\frac{p}{2}}|^2 + |u|^p \right) dx \leq C. \quad (2.35)$$

This immediately gives (2.22).

Furthermore we have the following corresponding inequalities to (2.26) (2.30):

$$\begin{aligned} & t \left(\|u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 \right) + \int_0^t \tau \|\nabla u_t\|^2 d\tau \\ & \leq C \int_0^t \tau \|u_t\|^2 d\tau + \int_0^t \left(\|u_t\|^2 + \|u_t\|_{L^2(\Gamma)}^2 \right) d\tau. \end{aligned} \quad (2.36)$$

$$\begin{aligned} & t^2 \|\nabla u_t\|^2 + \int_0^t \tau^2 \|\Delta u_t\|^2 d\tau + 2 \int_0^t \tau^2 \|u_{tt}\|_{L^2(\Gamma)}^2 d\tau \\ & \leq 2 \int_0^t \tau \|\nabla u_t\|^2 d\tau + 2C \int_0^t \tau^2 \|u_t\|_{H^1(\Omega)}^2 d\tau. \end{aligned} \quad (2.37)$$

By using the equivalent $H^1(\Omega)$ norm of u_t , it is easy to see that (2.31) holds as well.

Now for both cases, we consider the following elliptic boundary value problem

$$\begin{cases} \Delta u = u_t + f(u), \\ \partial_\nu u + \mu u + u_t|_{\Gamma} = 0, \end{cases} \quad (2.38)$$

according to the elliptic regularity theory and the trace theorem we have

$$\|u\|_{H^2(\Omega)} \leq C(\|u_t + f(u)\| + \|u_t\|_{H^{\frac{1}{2}}(\Gamma)} + \|u\|) \leq C(\|u_t\|_{H^1(\Omega)} + \|f(u)\| + \|u\|), \quad (2.39)$$

where C is certain positive constant depending only on Ω .

Combining the fact

$$\frac{2(n+2)}{n-2} < \frac{4n}{n-2} \quad \text{for } n \geq 3$$

with the result (2.21), assumption **(F2)** and Young's inequality yields that

$$\|f(u)\| \leq C. \quad (2.40)$$

By (2.31), (2.39), (2.40), for any $\sigma > 0$, we obtain

$$\|u\|_{H^2(\Omega)} \leq C_\sigma, \quad \forall t \geq \sigma \quad (2.41)$$

where C_σ is a constant depending on $\|u_0\|_{H^1(\Omega)}$, $\|u_0\|_{L^{\frac{4n}{n-2}}(\Omega)}$, $\|\gamma(u_0)\|_{L^{\frac{4n}{n-2}}(\Gamma)}$, σ and $|\Omega|$. \square

Remark 2.1. We consider the case $n \geq 3$ in the proof when we want to get the estimate of $\|f(u)\|$ and also $\|f'(u)\|_{L^n(\Omega)}$. For $n = 1, 2$, the proof is simpler because we can simply use $\|u_0\|_{H^1(\Omega)}$ to get the desired estimates due to the Sobolev imbedding inequality for $n = 1, 2$.

3 Proof of Theorem 1.2

The proof of Theorem 1.2 consists of several steps.

Step 1.

From (2.3) it is clear that the energy $E(u(t))$ is decreasing with respect to time. On the other hand, (2.8) or (2.12) implies that $E(u(t))$ is bounded from below. Thus we know that $E(u(t))$ serves as a Lyapunov functional. Besides, $\forall t > 0$, if $E(u) \equiv E(u_0)$ then $\frac{dE}{dt} \equiv 0$. This enables us to deduce that $u_t \equiv 0$, which means u is an equilibrium. The ω -limit set of $u_0 \in H^1(\Omega)$ is defined as follows:

$$\omega(u_0) = \{\psi(x) \in H^1(\Omega) : \exists \{t_n\}_{n=1}^\infty, t_n \rightarrow +\infty \text{ s. t. } u(x, t_n) \rightarrow \psi(x) \text{ in } H^1(\Omega)\}.$$

Then from the well-known results in the dynamic system (e.g., [27], Lemma I.1.1) it is easy to see that

Lemma 3.1. *The ω -limit set of u_0 is a non-empty compact connected subset in $H^1(\Omega)$. Furthermore, (i) it is invariant under the nonlinear semigroup $S(t)$ defined by the solution $u(x, t)$, i.e., $S(t)\omega(u_0) = \omega(u_0)$ for all $t \geq 0$. (ii) $E(u)$ is constant on $\omega(u_0)$. Moreover, $\omega(u_0)$ consists of equilibria.*

Step 2. In this step, we collect some results on the stationary problem.

Lemma 3.2. *Suppose that $\psi \in H^2(\Omega)$ is a strong solution to problem (1.5). Then ψ is a critical point of the functional $E(u)$ in $H^1(\Omega)$. Conversely, if ψ is a critical point of the functional $E(u)$ in $H^1(\Omega)$, then $\psi \in H^2(\Omega)$, and it is a strong solution to problem (1.5).*

Proof. If $\psi \in H^2(\Omega)$ satisfies (1.5), then for any $v \in H^1(\Omega)$, it follows from (1.5) that

$$\int_{\Omega} (-\Delta\psi + f(\psi))v dx = 0.$$

By integration by parts and the boundary condition in (1.5), we get

$$\int_{\Omega} (\nabla\psi \cdot \nabla v + f(\psi)v) dx + \mu \int_{\Gamma} \psi v dS = 0 \quad (3.1)$$

which, by a straightforward calculation, is just the following

$$\left. \frac{dE(\psi + \varepsilon v)}{d\varepsilon} \right|_{\varepsilon=0} = 0.$$

Thus, ψ is a critical point of $E(u)$. Conversely, if ψ is a critical point of $E(u)$ in $H^1(\Omega)$, then (3.1) is satisfied. By the assumption **(F2)** on subcritical growth and the bootstrap argument, $\psi \in H^2(\Omega)$ follows. \square

The following lemma claims that problem (1.5) admits at least a strong solution.

Lemma 3.3. *The functional $E(u)$ has at least a minimizer $v \in H^1(\Omega)$ such that*

$$E(v) = \inf_{u \in H^1(\Omega)} E(u).$$

In other words, problem (1.5) admits at least a strong solution.

Proof. From Section 2 we can see that $E(u)$ is bounded from below on $H^1(\Omega)$. Therefore, there is a minimizing sequence $u_n \in H^1(\Omega)$ such that

$$E(u_n) \rightarrow \inf_{u \in H^1} E(u). \quad (3.2)$$

On the other hand, $E(u)$ can be written in the form:

$$E(u) = \frac{1}{2} \|\nabla u\|^2 + \frac{\mu}{2} \|u\|_{L^2(\Gamma)}^2 + \mathcal{F}(u) \quad (3.3)$$

with

$$\mathcal{F}(u) = \int_{\Omega} F(u) dx, \quad (3.4)$$

it follows from (2.8) ((2.12) for $\mu = 0$) that u_n is bounded in $H^1(\Omega)$. It turns out from the weak compactness that there is a subsequence, still denoted by u_n , such that u_n weakly converges to v in $H^1(\Omega)$. Thus, $v \in H^1(\Omega)$. We infer from the Sobolev imbedding theorem that the imbedding $H^1(\Omega) \hookrightarrow L^\gamma(\Omega)$ ($1 \leq \gamma < \frac{n+2}{n-2}$) is compact. As a result, u_n strongly converges to v in $L^\gamma(\Omega)$. It turns out from the assumption **(F2)** that $\mathcal{F}(u_n) \rightarrow \mathcal{F}(v)$. Since $\|u\|_{H^1}^2$ is weakly lower semi-continuous, it follows from (3.2) that $E(v) = \inf_{u \in H^1} E(u)$. From the assumption **(F2)** on subcritical growth, by the bootstrap argument and elliptic regularity theorem, this weak solution is also a strong solution in $H^2(\Omega)$. Furthermore, we can also use the bootstrap argument to show that $v \in L^\infty(\Omega)$. The proof is completed. \square

Remark 3.1. *Under assumption **(F2)**, by a further bootstrap argument, we can show that ψ is also a classical solution.*

Step 3. Define

$$\mathcal{D} := \{u \in H^2(\Omega) \mid \partial_\nu u + \mu u|_{\Gamma} = 0\}.$$

Let ψ be a fixed critical point of $E(u)$. In what follows we are going to establish the generalized Łojasiewicz–Simon inequality which extends the original one by Simom [26] for the second order nonlinear parabolic equation subject to the Dirichlet boundary condition. Similar inequalities have been derived in our recent papers [30–32] to prove the convergence to equilibrium for various evolution equations (systems) subject to the dynamical boundary conditions.

Lemma 3.4. *Let ψ be a critical point of $E(u)$. Then there exist constants $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ depending on ψ such that for any $u \in H^2(\Omega)$ satisfying $\|u - \psi\|_{H^1(\Omega)} < \beta$ and $\|\partial_\nu u + \mu u\|_{L^2(\Gamma)} < \beta$, we have*

$$\| -\Delta u + f(u) \|_{(H^1(\Omega))'} + \|\partial_\nu u + \mu u\|_{L^2(\Gamma)} \geq |E(u) - E(\psi)|^{1-\theta}. \quad (3.5)$$

Proof. Let

$$M(u) = -\Delta u + f(u). \quad (3.6)$$

Then $M(u)$ maps $u \in H^2(\Omega)$ into $L^2(\Omega)$.

First, we claim that there exist constants $\theta' \in (0, \frac{1}{2})$ and $\beta_1 > 0$ depending on ψ , $\forall w \in \mathcal{D}$ (i.e., w satisfies the homogeneous Neumann or Robin boundary condition) satisfying $\|w - \psi\|_{H^1} < \beta_1$, we have

$$\|M(w)\|_{(H^1(\Omega))'} \geq C|E(w) - E(\psi)|^{1-\theta'}. \quad (3.7)$$

The above claim follows from the same argument as that in [14, 17] for the case of homogeneous Dirichlet boundary condition (see also [29]), so the details can be omitted here. However, we are now dealing with the nonhomogeneous boundary condition. As a result, we have to extend the above result from functions in \mathcal{D} to $H^2(\Omega)$. Hereafter we always keep in mind that ψ is an equilibrium satisfying $\partial_\nu \psi + \mu \psi = 0$ on Γ . As before we consider two cases.

Case 1 $\mu = 1$. For any $u \in H^2(\Omega)$, we consider the following Robin Problem:

$$\begin{cases} \Delta w = \Delta u, & x \in \Omega, \\ \partial_\nu w + w = 0, & x \in \Gamma. \end{cases}$$

It has a unique solution w belonging to \mathcal{D} . On the other hand, we introduce the Robin map $R : H^s(\Gamma) \rightarrow H^{s+(3/2)}(\Omega)$ which is defined as follows:

$$Rp = q \iff \begin{cases} \Delta q = 0 & \text{in } \Omega, \\ \partial_\nu q + q = p & \text{on } \Gamma. \end{cases}$$

As shown in [5], R is continuous for $s \in \mathbb{R}$. Thus we have

$$\|w - u\|_{H^1(\Omega)} \leq C\|\partial_\nu u + u\|_{L^2(\Gamma)}, \quad (3.8)$$

where the constant C does not depend on u . As a result,

$$\|w - \psi\|_{H^1(\Omega)} \leq \|w - u\|_{H^1(\Omega)} + \|u - \psi\|_{H^1(\Omega)} \leq C\|\partial_\nu u + u\|_{L^2(\Gamma)} + \|u - \psi\|_{H^1(\Omega)}. \quad (3.9)$$

Then it is easy to see the fact that $u \in H^2(\Omega)$ being in the small neighbourhood of ψ in $H^1(\Omega)$ and $\|\partial_\nu u + u\|_{L^2(\Gamma)}$ being small imply that w stays in a small neighbourhood of ψ in $H^1(\Omega)$.

By direct calculation and the Sobolev imbedding theorem we can see that

$$\begin{aligned}
\|M(w)\|_{(H^1(\Omega))'} &\leq \|M(u)\|_{(H^1(\Omega))'} + \|f(u) - f(w)\|_{(H^1(\Omega))'} \\
&\leq \|M(u)\|_{(H^1(\Omega))'} + \left\| \int_0^1 f'(u + t(w - u))(u - w) dt \right\|_{(H^1(\Omega))'} \\
&\leq \|M(u)\|_{(H^1(\Omega))'} + \max_{0 \leq t \leq 1} \|f'(u + t(w - u))\|_{L^{\frac{n}{2}}(\Omega)} \|u - w\|_{L^{\frac{2n}{n-2}}(\Omega)} \\
&\leq \|M(u)\|_{(H^1(\Omega))'} + C \|u - w\|_{H^1(\Omega)} \\
&\leq \|M(u)\|_{(H^1(\Omega))'} + C \|\partial_\nu u + u\|_{L^2(\Gamma)}.
\end{aligned} \tag{3.10}$$

On the other hand, by the Newton-Leibniz formula and (3.8) we can get

$$\begin{aligned}
&|E(w) - E(u)| \\
&\leq \left| \int_0^1 \int_\Omega M(u + t(w - u))(u - w) dx dt \right| + \left| \int_0^1 \int_\Gamma (1 - t)(\partial_\nu u + u)(u - w) dS dt \right| \\
&\leq C (\|M(u)\|_{(H^1(\Omega))'} + \|\partial_\nu u + u\|_{L^2(\Gamma)}) \|\partial_\nu u + u\|_{L^2(\Gamma)} \\
&\leq C (\|M(u)\|_{(H^1(\Omega))'} + \|\partial_\nu u + u\|_{L^2(\Gamma)})^2.
\end{aligned} \tag{3.11}$$

Since

$$|E(w) - E(\psi)|^{1-\theta'} \geq |E(u) - E(\psi)|^{1-\theta'} - |E(w) - E(u)|^{1-\theta'}, \tag{3.12}$$

and $0 < \theta' < \frac{1}{2}$, $2(1 - \theta') - 1 > 0$, then from (3.7) and (3.10)–(3.11) we can conclude that

$$C(\|M(u)\|_{(H^1(\Omega))'} + \|\partial_\nu u + u\|_{L^2(\Gamma)}) \geq |E(u) - E(\psi)|^{1-\theta'},$$

for $\|u - \psi\|_{H^1(\Omega)} < \beta$, $\|\partial_\nu u + u\|_{L^2(\Gamma)} < \beta$, where $\beta > 0$ is chosen small enough such that $\|w - \psi\|_{H^1} < \beta_1$.

Next we choose ε , $0 < \varepsilon < \theta'$ and β smaller if necessary such that when $\|u - \psi\|_{H^1} < \beta$,

$$\frac{1}{C} |E(u) - E(\psi)|^{-\varepsilon} \geq 1. \tag{3.13}$$

Setting $\theta = \theta' - \varepsilon \in (0, \frac{1}{2})$, when $\|u - \psi\|_{H^1} < \beta$ and $\|\partial_\nu u + u\|_{L^2(\Gamma)} < \beta$, we finally have

$$\|M(u)\|_{(H^1(\Omega))'} + \|\partial_\nu u + u\|_{L^2(\Gamma)} \geq |E(u) - E(\psi)|^{1-\theta}. \tag{3.14}$$

which is exactly (3.5).

Case 2 $\mu = 0$. For any $u \in H^2(\Omega)$, we consider the following Neumann Problem:

$$\begin{cases} -\Delta w + w = -\Delta u + u, & x \in \Omega, \\ \partial_\nu w = 0, & x \in \Gamma. \end{cases}$$

It's easy to see that there exists a unique solution w belonging to \mathcal{D} and

$$\|w\|_{H^1(\Omega)} \leq C (\|u\|_{H^1(\Omega)} + \|\partial_\nu u\|_{L^2(\Gamma)}).$$

Again we can see that u being in the small neighbourhood of ψ in $H^1(\Omega)$ with $\|\partial_\nu u\|_{L^2(\Gamma)}$ small implies that w stays in a small neighbourhood of ψ in $H^1(\Omega)$.

Furthermore, by the energy estimate, we have

$$\|w - u\|_{H^1(\Omega)} \leq C \|\partial_\nu u\|_{L^2(\Gamma)}. \quad (3.15)$$

In a way similar to Case 1, the following estimates follow:

$$\begin{aligned} & \|M(w)\|_{(H^1(\Omega))'} \\ & \leq \|M(u)\|_{(H^1(\Omega))'} + \|(f(u) - u) - (f(w) - w)\|_{(H^1(\Omega))'} \\ & \leq \|M(u)\|_{(H^1(\Omega))'} + \max_{0 \leq t \leq 1} \|f'(u + t(w - u))\|_{L^{\frac{n}{2}}(\Omega)} \|u - w\|_{L^{\frac{2n}{n-2}}(\Omega)} + \|u - w\|_{(H^1(\Omega))'} \\ & \leq \|M(u)\|_{(H^1(\Omega))'} + C \|u - w\|_{H^1(\Omega)} \\ & \leq \|M(u)\|_{(H^1(\Omega))'} + C \|\partial_\nu u\|_{L^2(\Gamma)}. \end{aligned} \quad (3.16)$$

On the other hand, by the Newton-Leibniz formula and (3.15), we can get

$$\begin{aligned} & |E(w) - E(u)| \\ & \leq \left| \int_0^1 \int_\Omega M(u + t(w - u))(u - w) dx dt \right| + \left| \int_0^1 \int_\Gamma (1 - t) \partial_\nu u (u - w) dS dt \right| \\ & \leq C (\|M(u)\|_{(H^1(\Omega))'} + \|\partial_\nu u\|_{L^2(\Gamma)})^2. \end{aligned} \quad (3.17)$$

Then arguing as in Case 1, we can get the required corresponding result for $\mu = 0$.

Thus, the lemma is proved. \square

Following the idea in [32] (see also [30]) we can easily extend the previous lemma in such a way that we only need the smallness of $\|u - \psi\|_{H^1(\Omega)}$,

Lemma 3.5 (Generalized Łojasiewicz–Simon Inequality). *Let ψ be a critical point of $E(u)$. Then there exist constants $\theta \in (0, \frac{1}{2})$ and $\beta_0 > 0$ depending on ψ such that for $\forall u \in H^2(\Omega)$, $\|u - \psi\|_{H^1(\Omega)} < \beta_0$, we have*

$$\|-\Delta u + f(u)\|_{(H^1(\Omega))'} + \|\partial_\nu u + \mu u\|_{L^2(\Gamma)} \geq |E(u) - E(\psi)|^{1-\theta}. \quad (3.18)$$

Proof. Let $\beta > 0$ and $\theta \in (0, \frac{1}{2})$ be the constants appearing in Lemma 3.4.

We consider the following two cases:

- (i) $\|u - \psi\|_{H^1(\Omega)} < \beta$ and $\|\partial_\nu u + \mu u\|_{L^2(\Gamma)} < \beta$. Then from Lemma 3.4 we are done.
- (ii) $\|u - \psi\|_{H^1(\Omega)} < \beta$ but $\|\partial_\nu u + \mu u\|_{L^2(\Gamma)} \geq \beta$. By Sobolev imbedding $H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, there exist $\tilde{\beta} > 0$ depending on ψ such that for any u satisfying $\|u - \psi\|_{H^1(\Omega)} < \tilde{\beta}$,

$$|E(u) - E(\psi)|^{1-\theta} < \beta. \quad (3.19)$$

Thus, we have

$$\begin{aligned} \| -\Delta u + f(u) \|_{(H^1(\Omega))'} + \|\partial_\nu u + \mu u\|_{L^2(\Gamma)} &\geq \|\partial_\nu u + \mu u\|_{L^2(\Gamma)} \\ &\geq \beta \\ &> |E(u) - E(\psi)|^{1-\theta}. \end{aligned} \quad (3.20)$$

Finally by setting $\beta_0 = \min\{\beta, \tilde{\beta}\}$, the lemma is proved. \square

Step 4. After the previous preparations, we now proceed to finish the proof of Theorem 1.2, following a simplified argument introduced in [17] in which the key observation is that after certain time t_0 , the solution u will fall into the small neighborhood of a certain equilibrium $\psi(x)$, and stay there forever.

First, it is easy to see that

$$\|u_t\| \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

From Step 1, there is a sequence $\{t_n\}_{n \in \mathbb{N}}$, $t_n \rightarrow +\infty$ such that

$$u(x, t_n) \rightarrow \psi(x), \quad \text{in } H^1(\Omega), \quad (3.21)$$

where $\psi(x) \in \omega(u_0)$ is a certain equilibrium. On the other hand, it follows from (2.3) that $E(u)$ is decreasing in time and

$$\lim_{n \rightarrow +\infty} E(t_n) = E(\psi). \quad (3.22)$$

We now consider all possibilities.

(1). If there is a $t_0 > 0$ such that at this time $E(u) = E(\psi)$, then for all $t > t_0$, we deduce from (2.3) that u is independent of t . Since $u(x, t_n) \rightarrow \psi$, the theorem is proved.

(2). If there is $t_0 > 0$ such that for all $t \geq t_0$, u satisfies the condition of Lemma 3.5, i.e., $\|u - \psi\|_{H^1(\Omega)} < \beta_0$, then for $\theta \in (0, \frac{1}{2})$ introduced in Lemma 3.5, we have

$$\frac{d}{dt}(E(u) - E(\psi))^\theta = \theta(E(u) - E(\psi))^{\theta-1} \frac{dE(u)}{dt}. \quad (3.23)$$

Combining it with (3.18), (2.3) yields

$$\frac{d}{dt}(E(u) - E(\psi))^\theta + \frac{\theta}{2} (\| -\Delta u + f(u) \| + \| u_t \|_{L^2(\Gamma)}) \leq 0. \quad (3.24)$$

Integrating from t_0 to t , we get

$$(E(u) - E(\psi))^\theta + \frac{\theta}{2} \int_{t_0}^t (\| -\Delta u + f(u) \| + \| u_t \|_{L^2(\Gamma)}) d\tau \leq (E(u(t_0)) - E(\psi))^\theta. \quad (3.25)$$

Since $E(u) - E(\psi) \geq 0$, we have

$$\int_{t_0}^t (\| -\Delta u + f(u) \| + \| u_t \|_{L^2(\Gamma)}) d\tau < +\infty, \quad (3.26)$$

which implies that for all $t \geq t_0$,

$$\int_{t_0}^t \|u_t\| d\tau \leq C. \quad (3.27)$$

This easily yields that as $t \rightarrow +\infty$, $u(x, t)$ converges in $L^2(\Omega)$. Since the orbit is compact in $H^1(\Omega)$, we can deduce from uniqueness of limit that (1.4) holds, and the theorem is proved.

(3). It follows from (3.21) that for any $\varepsilon > 0$ with $\varepsilon < \beta_0$, there exists an integer N such that when $n \geq N$,

$$\| u(\cdot, t_n) - \psi \| \leq \| u(\cdot, t_n) - \psi \|_{H^1(\Omega)} < \frac{\varepsilon}{2}, \quad (3.28)$$

$$\frac{1}{\theta}(E(u(t_n)) - E(\psi))^\theta < \frac{\varepsilon}{4}. \quad (3.29)$$

Define

$$\bar{t}_n = \sup \{ t > t_n \mid \| u(\cdot, s) - \psi \|_{H^1(\Omega)} < \beta_0, \forall s \in [t_n, t] \} \quad (3.30)$$

It follows from (3.28) and continuity of the orbit in $H^1(\Omega)$ for $t > 0$ that $\bar{t}_n > t_n$ for all $n \geq N$. Then there are two possibilities:

(i). If there exists $n_0 \geq N$ such that $\bar{t}_{n_0} = +\infty$, then from the previous discussions in (1)(2), the theorem is proved.

(ii) Otherwise, for all $n \geq N$, we have $t_n < \bar{t}_n < +\infty$, and for all $t \in [t_n, \bar{t}_n]$, $E(\psi) < E(u(t))$. Then from (3.25) with t_0 being replaced by t_n , and t being replaced by \bar{t}_n we deduce that

$$\int_{t_n}^{\bar{t}_n} \|u_t\| d\tau \leq \frac{2}{\theta}(E(u(t_n)) - E(\psi))^\theta < \frac{\varepsilon}{2}. \quad (3.31)$$

Thus, it follows that

$$\|u(\bar{t}_n) - \psi\| \leq \|u(t_n) - \psi\| + \int_{t_n}^{\bar{t}_n} \|u_t\| d\tau < \varepsilon \quad (3.32)$$

which implies that when $n \rightarrow +\infty$,

$$u(\bar{t}_n) \rightarrow \psi, \quad \text{in } L^2(\Omega). \quad (3.33)$$

Since $\bigcup_{t \geq \delta} u(t)$ is relatively compact in $H^1(\Omega)$, there exists a subsequence of $\{u(\bar{t}_n)\}$, still denoted by $\{u(\bar{t}_n)\}$ converging to ψ in $H^1(\Omega)$. Namely, when n is sufficiently large,

$$\|u(\bar{t}_n) - \psi\|_{H^1(\Omega)} < \beta_0, \quad (3.34)$$

which contradicts the definition of \bar{t}_n that $\|u(\cdot, \bar{t}_n) - \psi\|_{H^1(\Omega)} = \beta_0$.

Thus, Theorem 1.2 is proved.

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